Lepton mixing and CP violation phase in the 3-3-1 model with neutral leptons based on T_{13} flavor symmetry

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We study a 3-3-1 model based on non-Abelian discrete symmetry group T_{13} which accommodates lepton mixing with non-zero θ_{13} and CP violation phase. The neutrinos get small masses and mixing with CP violation phase from $SU(3)_L$ antisextets which are all in triplets under T_{13}. If both breakings $T_{13} \rightarrow Z_3$ and $Z_3 \rightarrow \{\text{Identity}\}$ are taken palce in neutrino sector and T_{13} is broken into Z_3 in lepton sector, the realistic neutrino mixing form is obtained as a natural consequence of P_l and T_{13} symmetries. The model predicts the lepton mixing with non-zero θ_{13}, and also gives a remarkable prediction of Dirac CP violation $\delta_{CP} = 292.5^\circ$ in the normal spectrum, and $\delta_{CP} = 303.161^\circ$ in the inverted spectrum which is still missing in the neutrino mixing matrix. There exist some regions of model parameters that can fit the experimental data in 2014 on neutrino masses and mixing without perturbation.

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I. INTRODUCTION

The Standard Model (SM) is one of the most successful and thoroughly tested theories in the elementary particle physics field, however, the origin of flavor structure, masses and mixings between generations of matter particles are unknown yet. Many experiments show that neutrinos have tiny masses and their mixing is sill mysterious [1, 2]. The neutrino mass and mixing is one of the most important evidence of beyond Standard Model physics. Among the possible extensions of SM, the 3-3-1 models, which encompass a class of models based on the gauge group $\mathrm{SU}(3)_C \otimes \mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$ [3–19], have revealed interesting features. The first

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one is that the requirement of anomaly cancelation together with that of asymptotic freedom of QCD implies that the number of generations must necessarily be equal to the number of colors, hence giving an explanation for the existence of three generations. Furthermore, quark generations should transform differently under the action of $SU(3)_L$. In particular, two quark generations should transform as triplets, one as an antitriplet.

A fundamental relation holds among some of the generators of the subgroups in the models [20, 21]:

$$Q = T_3 + \beta T_8 + X \tag{1}$$

where Q indicates the electric charge, T_3 and T_8 are two of the $SU(3)$ generators and X is the generator of $U(1)_X$. β is a key parameter that defines a specific variant of the model.

There are two typical variants of the 3-3-1 models as far as lepton sectors are concerned. In the minimal version, three $\mathrm{SU}(3)_L$ lepton triplets are (ν_L, l_L, l^c_R), where l_R are ordinary right-handed charged-leptons [3–7] . In the second version, the third components of lepton triplets are the right-handed neutrinos, (ν_L, l_L, ν^c_R) [8–13]. To have a model with the realistic neutrino mixing matrix, we should consider another variant of the form (ν_L, l_L, N^c_R) where N_R are three new fermion singlets under SM symmetry with vanishing lepton-numbers [22–25] .

The recent data have provided the evidence for a non-vanishing value of the smallest mixing angle θ_{13} [26] but the tri-bimaximal form for explaining the lepton mixing scheme was first proposed by Harrison-Perkins-Scott (HPS), which apart from the phase redefinitions, is given by [27–30]

$$U_{\mathrm{HPS}} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \tag{2}$$

can be considered as a good approximation for the recent neutrino experimental data. In fact, the absolute values of the entries of the lepton mixing matrix U_{PMNS} approximately are given by [31]

$$|U_{\mathrm{PMNS}}| = \begin{pmatrix} 0.795-0.846 & 0.513-0.585 & 0.126-0.178 \\ 0.205-0.543 & 0.416-0.730 & 0.579-0.808 \\ 0.215-0.548 & 0.409-0.725 & 0.567-0.800 \end{pmatrix}, \tag{3}$$

while the most recent data in PDG2014 [32] imply[1]:

$$\begin{aligned}
&\sin^2(2\theta_{12}) = 0.846 \pm 0.021,\ \Delta m^2_{21} = (7.53 \pm 0.18) \times 10^{-5}\mathrm{eV}^2, \\
&\sin^2(2\theta_{13}) = (9.3 \pm 0.8) \times 10^{-2}, \\
&\sin^2(2\theta_{23}) = 0.999^{+0.001}_{-0.018},\ \ \Delta m^2_{32} = (2.44 \pm 0.06) \times 10^{-3}\mathrm{eV}^2,\ (\mathrm{NH}), \\
&\sin^2(2\theta_{23}) = 1.000^{+0.000}_{-0.017},\ \ \Delta m^2_{32} = (2.52 \pm 0.07) \times 10^{-3}\mathrm{eV}^2,\ (\mathrm{IH}),
\end{aligned} \tag{4}$$

with a slight deviation from Tri-bimaximal mixing form given in (2). These large neutrino mixing angles are completely different from the quark mixing ones defined by the Cabibbo-Kobayashi-Maskawa (CKM) matrix [33, 34] . This has stimulated work on flavor symmetries and non-Abelian discrete symmetries are considered to be the most attractive candidate to formulate dynamical principles that can lead to the flavor mixing patterns for quarks, lepton and some other issues related to the flavor physics. In order to overcome these problems, plenty of models based on the principle of symmetry, flavor symmetry, have been discussed. Among them, non-Abelian discrete symmetries are well discussed as plausible possibilities. In particular, the fact that the lepton mixing matrix (U_{PMNS}) shows very good agreement with the tri-bimaximal form [27–30] implies that flavor structure is originated from a symmetry.

There are many recent models based on the non-Abelian discrete symmetries, such as A_4 [35–52] , A_5 [53–65] , S_3[66–107] , S_4 [108–136] , D_4 [137–148] , D_5 [149, 150] , T' [151–155] and so forth. In the context of the 3-3-1 model, the non-abelian discrete symmetries A_4, S_3, S_4, D_4, T_7 have been explored [22–25, 156–158], however, T_{13} symmetry has not been considered before in this kind of the model. For the similar works on T_{13}, let us call the reader's attention to Refs. [160–162]. In Ref. [23] we have studied the 3-3-1 model with neutral fermions based on S_4 group, in which most of the Higgs multiplets are in triplets under S_4 except χ lying in a singlet, and the exact tribimaximal form [27–30] is obtained. As we know, the recent considerations have implied $\theta_{13} \neq 0$ [35–52, 66–136], but small as given in (4). This problem has been improved in Refs. [24, 25] by adding new Higgs multiplets and using the perturbation theory up to the first order.

CP violation plays a crucial role in our understanding of the observed baryon asymmetry of the Universe (BAU) [163]. In the SM, CP symmetry is violated due to a complex phase

[1] In this paper, NH and IH stand for the normal and inverted mass hierarchies, respectively.

in the CKM matrix [33, 34]. However, since the extent of CP violation in the SM is not enough for achieving the observed BAU, we need new source of CP violation for successful BAU. On the other hand, CP violations in the lepton sector are imperative if the BAU could be realized through leptogenesis. So, any hint or observation of the leptonic CP violation can strengthen our belief in leptogenesis [163]. The violation of the CP symmetry is a very important ingredient of any dynamical mechanism which intends to explain both low energy CP violation and the baryon asymmetry. Renormalizable gauge theories are based on the spontaneous symmetry breaking mechanism, and it is natural to have the spontaneous CP violation as an integral part of that mechanism. Determining all possible sources of CP violation is a fundamental challenge for high energy physics. In theoretical and economical viewpoints, the spontaneous CP breaking necessary to generate the baryon asymmetry and leptonic CP violation at low energies brings us to a common source which comes from the phase of the scalar field responsible for the spontaneous CP breaking at a high energy scale [163].

In this work, we investigate another choice with the Frobenius group T_{13}, which is isomorphic to $Z_{13} \rtimes Z_3$. The T_{13} group is a subgroup of $SU(3)_L$, and known as the minimal non-Abelian discrete group having two complex triplets as the irreducible representations. T_{13} contains two complex irreducible representations $\underline{3}_1, \underline{3}_2$ and three singlets $\underline{1}_0$, $\underline{1}_1$, $\underline{1}_2$. This feature is useful to separate the three families of fermions from the others as requirement of the $3-3-1$ models. Namely, in this work, three inequivalent singlet representations $\underline{1}_0, \underline{1}_1$ and $\underline{1}_2$ of T_{13} play a crucial role in consistently reproducing fermion masses and mixings which allow to naturally accommodate the three right-handed charged-leptons, three left-handed components of the ordinary quarks and the right-handed components of the exotic quarks. On the other hand, three left- handed leptons and three right-handed components of the ordinary up-quarks and down-quarks are accommodated in two complex irreducible representations which can generate small masses for neutrinos since the good feature of tensor products $\underline{\bar{3}}_1 \otimes \underline{3}_1$ under T_{13}: $\underline{\bar{3}}_1 \otimes \underline{3}_1 = 1_0 \oplus 1_1 \oplus 1_2 \oplus \underline{3}_2 \oplus \underline{\bar{3}}_2$.

A brief of the theory of T_{13} group is given in Appendix A. Two important tensor products

of T_{13} used to construct the Yukawa interactions that responsible for fermion masses are

$$\begin{aligned}
\underline{\bar{3}}_1 \otimes \underline{3}_1 &= \underline{3}_1 \otimes \underline{\bar{3}}_1 = 1_0(11+22+33) \oplus 1_1(11+\omega 22+\omega^2 33) \\
&\oplus 1_2(11+\omega^2 22+\omega 33) \oplus \underline{3}_2(21,32,13) \oplus \underline{\bar{3}}_2(12,23,31), \qquad (5)\\
\underline{\bar{3}}_2 \otimes \underline{3}_2 &= \underline{3}_2 \otimes \underline{\bar{3}}_2 = 1_0(11+22+33) \oplus 1_1(11+\omega 22+\omega^2 33) \\
&\oplus 1_2(11+\omega^2 22+\omega 33) \oplus \underline{3}_1(23,31,12) \oplus \underline{\bar{3}}_1(32,13,21).
\end{aligned}$$

The rest of this work is organized as follows. In Sec. II and III we present the necessary elements of the 3-3-1 model with T_{13} symmetry as well as introducing necessary Higgs fields responsible for the charged lepton masses. In Sec. IV, we discuss on quark sector. Sec. V is devoted for the neutrino mass and mixing. We summarize our results and make conclusions in the section VI. Appendix A presents a brief of the T_{13} theory.

II. FERMION CONTENT

In the model under consideration, the gauge symmetry is given by $\mathrm{SU}(3)_C \otimes \mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$, where the electroweak factor $\mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$ is extended from those of the SM while the strong interaction sector is retained. Each lepton family includes a new lepton singlet carrying no lepton-number (N_R) and is arranged under the $\mathrm{SU}(3)_L$ symmetry as a triplet (ν_L, l_L, N^c_R) and a singlet l_R. We consider the version with $\beta = -\frac{1}{\sqrt{3}}$ in Eq.(1), hence, there is no exotic electric charges in the fundamental fermion, scalar and adjoint gauge boson representations. Since the particles in the lepton triplet have different lepton numbers (0 and 1), so the lepton number in the model does not commute with the gauge symmetry unlike the SM. Therefore, it is better to work with a new conserved charge $\mathcal{L}$ commuting with the gauge symmetry and related to the ordinary lepton number (L) by diagonal matrices [22–25, 164]

$$L = \frac{2}{\sqrt{3}} T_8 + \mathcal{L}. \qquad (6)$$

The lepton charge arranged in this way [i.e. $L(N_R) = 0$ as assumed] is in order to prevent unwanted interactions due to $\mathrm{U}(1)_{\mathcal{L}}$ symmetry and breaking to obtain the consistent lepton and quark spectra. By this embedding, exotic quarks U, D as well as new non-Hermitian gauge bosons X^0, $Y^{\pm}$ possess lepton charges as of the ordinary leptons, $L(D) = -L(U) = L(X^0) = L(Y^-) = 1$.

In the model under consideration, we put three left- handed leptons, three right-handed components of the ordinary up-quarks in $\underline{3}_1$ and three right-handed components of the ordinary down-quarks in $\underline{\bar{3}}_1$, while three right-handed charged-leptons, three left-handed components of the ordinary quarks and three right-handed components of the exotic quarks are in the singlets. Under the $[\mathrm{SU}(3)_L, \mathrm{U}(1)_X, \mathrm{U}(1)_{\mathcal{L}}, \underline{T}_{13}]$ symmetries as proposed, the fermions of the model, respectively, transform as follows

$$\begin{aligned}
\psi_L \equiv& \ \psi_{1,2,3L} = \left(\nu_{1,2,3L} \quad l_{1,2,3L} \quad N^c_{1,2,3R}\right)^T \sim [3, -1/3, 2/3, \underline{3}_1], \\
l_{1R} \sim& \ [1, -1, 1, \underline{1}_0], \ \ l_{2R} \sim [1, -1, 1, \underline{1}_1], \ \ l_{3R} \sim [1, -1, 1, \underline{1}_2], \\
Q_{1L} \equiv& \ (d_{1L} \quad -u_{1L} \quad D_{1L})^T \sim [3^*, 0, 1/3, \underline{1}_1], \\
Q_{2L} \equiv& \ (d_{2L} \quad -u_{2L} \quad D_{2L})^T \sim [3^*, 0, 1/3, \underline{1}_2], \\
Q_{3L} =& \ (u_{3L} \quad d_{3L} \quad U_L)^T \sim [3, 1/3, -1/3, \underline{1}_0], \\
u_{iR} \sim& \ [1, 2/3, 0, \underline{3}_1], \quad d_{iR} \sim [1, -1/3, 0, \underline{\bar{3}}_1], \\
D_{1R} \sim& \ [1, -1/3, 1, \underline{1}_2], \ D_{2R} \sim [1, -1/3, 1, \underline{1}_1], \ U_R \sim [1, 2/3, -1, \underline{1}_0].
\end{aligned} \tag{7}$$

where the subscript numbers on fields indicate to respective families which also in order define components of their T_{13} multiplets. In the following, we consider possibilities of generating the masses for the fermions. The scalar multiplets needed for the purpose are also introduced.

III. CHARGED-LEPTON MASSES

The charged-lepton masses arise from the couplings of $\bar{\psi}_L l_{iR}$ $(i = 1, 2, 3)$ to scalars, where $\bar{\psi}^c_L \psi_L$ transforms as $3^* \oplus 6$ under $\mathrm{SU}(3)_L$ and $\underline{3}_1$ under T_{13}. To generate masses for the charged leptons, we need one $SU(3)_L$ Higgs triplets put in $\underline{3}_1$ under T_{13},

$$\phi = \left(\phi^+_1 \quad \phi^0_2 \quad \phi^+_3\right)^T \sim [3, 2/3, -1/3, \underline{3}_1]. \tag{8}$$

The Yukawa interactions are

$$\begin{aligned}
-\mathcal{L}_l =& \ h_1(\bar{\psi}_L \phi)_{\underline{1}_0} l_{1R} + h_2(\bar{\psi}_L \phi)_{\underline{1}_2} l_{2R} + h_3(\bar{\psi}_L \phi)_{\underline{1}_1} l_{3R} + H.c \\
=& \ h_1(\bar{\psi}_{1L}\phi_1 + \bar{\psi}_{2L}\phi_2 + \bar{\psi}_{3L}\phi_3) l_{1R} \\
+& \ h_2(\bar{\psi}_{1L}\phi_1 + \omega^2\bar{\psi}_{2L}\phi_2 + \omega\bar{\psi}_{3L}\phi_3) l_{2R} \\
+& \ h_3(\bar{\psi}_{1L}\phi_1 + \omega\bar{\psi}_{2L}\phi_2 + \omega^2\bar{\psi}_{3L}\phi_3) l_{3R} + H.c.
\end{aligned} \tag{9}$$

Following the potential minimization condition, we have the followings alignments:

(1) The first alignment: $\langle\phi_1\rangle \neq \langle\phi_2\rangle \neq \langle\phi_3\rangle$ then T_{13} is completely broken.

(2) The second alignment: $0 \neq \langle\phi_1\rangle \neq \langle\phi_2\rangle = \langle\phi_3\rangle \neq 0$ or $0 \neq \langle\phi_1\rangle = \langle\phi_3\rangle \neq \langle\phi_2\rangle \neq 0$ or $0 \neq \langle\phi_1\rangle = \langle\phi_2\rangle \neq \langle\phi_3\rangle \neq 0$ then T_{13} is completely broken.

(3) The third alignment: $\langle\phi_1\rangle = \langle\phi_2\rangle = \langle\phi_3\rangle \neq 0$ then T_{13} is broken into Z_3 that consists of the elements $\{e, b, b^2\}$.

(4) The fourth alignment: $0 = \langle\phi_1\rangle \neq \langle\phi_2\rangle \neq \langle\phi_3\rangle \neq 0$ or $0 = \langle\phi_2\rangle \neq \langle\phi_1\rangle \neq \langle\phi_3\rangle \neq 0$ or $0 \neq \langle\phi_1\rangle \neq \langle\phi_2\rangle \neq \langle\phi_3\rangle = 0$ then T_{13} is completely broken.

(5) The fifth alignment: $0 = \langle\phi_1\rangle \neq \langle\phi_2\rangle = \langle\phi_3\rangle \neq 0$ or $0 = \langle\phi_2\rangle \neq \langle\phi_1\rangle = \langle\phi_3\rangle \neq 0$ or $0 \neq \langle\phi_1\rangle = \langle\phi_2\rangle \neq \langle\phi_3\rangle = 0$ then T_{13} is completely broken.

(6) The sixth alignment: $0 = \langle\phi_1\rangle = \langle\phi_2\rangle \neq \langle\phi_3\rangle$ or $0 = \langle\phi_1\rangle = \langle\phi_3\rangle \neq \langle\phi_2\rangle$ or $0 \neq \langle\phi_1\rangle \neq \langle\phi_2\rangle = \langle\phi_3\rangle = 0$ then T_{13} is completely broken.

Theoretically, a possibility that tribimaximal mixing matrix U_{HPS} in (2) can be decomposed into only two independent rotations may provide a hint for some underlying structure in the lepton sector, such as

$$U_{\mathrm{HPS}} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \end{pmatrix} \cong U_L^+ U_\nu, \qquad (10)$$

where $\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2$.

To obtain charged - lepton mixing satisfying (10), in this work we impose only the breaking $T_{13} \to Z_3$ in charged lepton sector, and this happens with the third alignment as above, i.e, $\langle\phi\rangle = (\langle\phi_1\rangle, \langle\phi_1\rangle, \langle\phi_1\rangle)$ under T_{13}, where

$$\langle\phi_1\rangle = (0 \quad v \quad 0)^T. \qquad (11)$$

The mass Lagrangian of the charged leptons takes the form:

$$\begin{aligned} -\mathcal{L}_l^{mass} &= h_1 v \bar{l}_{1L} l_{1R} + h_2 v \bar{l}_{1L} l_{2R} + h_3 v \bar{l}_{1L} l_{3R} \\ &+ h_1 v \bar{l}_{2L} l_{1R} + h_2 \omega^2 v \bar{l}_{2L} l_{2R} + h_3 \omega v \bar{l}_{2L} l_{3R} \\ &+ h_1 v \bar{l}_{3L} l_{1R} + h_2 \omega v \bar{l}_{3L} l_{2R} + h_3 \omega^2 v \bar{l}_{3L} l_{3R} + H.c. \end{aligned} \qquad (12)$$

We can rewite the Lagrangian (12) in the matrix form as follows

$$-\mathcal{L}_l^{\mathrm{mass}} = (\bar{l}_{1L}, \bar{l}_{2L}, \bar{l}_{3L}) M_l (l_{1R}, l_{2R}, l_{3R})^T + H.c, \tag{13}$$

where

$$M_l = \begin{pmatrix} h_1 v & h_2 v & h_3 v \\ h_1 v & h_2 v\omega^2 & h_3 v\omega \\ h_1 v & h_2 v\omega & h_3 v\omega^2 \end{pmatrix}. \tag{14}$$

The mass matrix M_l is then diagonalized,

$$U_L^\dagger M_l U_R = \begin{pmatrix} \sqrt{3}h_1 v & 0 & 0 \\ 0 & \sqrt{3}h_2 v & 0 \\ 0 & 0 & \sqrt{3}h_3 v \end{pmatrix} \equiv \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}, \tag{15}$$

where

$$U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \quad U_R = 1. \tag{16}$$

The charged lepton Yukawa couplings $h_{1,2,3}$ are defined as follows:

$$h_1 = \frac{m_e}{\sqrt{3}v}, \quad h_2 = \frac{m_\mu}{\sqrt{3}v}, \quad h_3 = \frac{m_\tau}{\sqrt{3}v}. \tag{17}$$

The experimental values for masses of the charged leptons are given in [32]:

$$m_e \simeq 0.51099\,\mathrm{MeV}, \quad m_\mu = 105.65837\ \mathrm{MeV}, \quad m_\tau = 1776.82\,\mathrm{MeV} \tag{18}$$

It follows that $h_1 \ll h_2 \ll h_3$. On the other hand, if we choose the VEV $v \sim 100GeV$ then

$$h_1 \sim 10^{-6}, \ \ h_2 \sim 10^{-4}, \ \ h_3 \sim 10^{-2}, \tag{19}$$

i.e, in the model under consideration, the hierarchy between the masses for charged-leptons can be achieved if there exists a hierarchy between Yukawa couplings $h_i\,(i = 1, 2, 3)$ in charged-lepton sector as given in (19).

IV. QUARK MASSES

To generate masses for quarks with a minimal Higgs content, we additionally introduce the following Higgs triplets

$$\begin{aligned} \chi &= \begin{pmatrix}\chi_1^0 & \chi_2^- & \chi_3^0\end{pmatrix}^T \sim [3,-1/3,2/3,\underline{1}_0], \\ \eta &= \begin{pmatrix}\eta_1^0 & \eta_2^- & \eta_3^0\end{pmatrix}^T \sim [3,-1/3,-1/3,\underline{3}_1]. \end{aligned} \tag{20}$$

The Yukawa interactions are

$$\begin{aligned} -\mathcal{L}_q &= f_3(\bar{Q}_{3L}\chi)_{\underline{1}_0}U_R + f_1(\bar{Q}_{1L}\chi^*)_{\underline{1}_1}D_{1R} + f_2(\bar{Q}_{2L}\chi^*)_{\underline{1}_2}D_{2R} \\ &+ h_1^d\bar{Q}_{1L}(\eta^* d_{iR})_{\underline{1}_2} + h_2^d\bar{Q}_{2L}(\eta^* d_{iR})_{\underline{1}_1} + h_3^d\bar{Q}_{3L}(\phi d_{iR})_{\underline{1}_0} \\ &+ h_1^u\bar{Q}_{1L}(\phi^* u_{iR})_{\underline{1}_2} + h_2^u\bar{Q}_{2L}(\phi^* u_{iR})_{\underline{1}_1} + h_3^u\bar{Q}_{3L}(\eta u_{iR})_{\underline{1}_0} + H.c. \\ &= f_3(\bar{Q}_{3L}\chi)_{\underline{1}_0}U_R + f_1(\bar{Q}_{1L}\chi^*)_{\underline{1}_1}D_{1R} + f_2(\bar{Q}_{2L}\chi^*)_{\underline{1}_2}D_{2R} \\ &+ h_1^d\bar{Q}_{1L}(\eta_1^* d_{1R} + \omega^2\eta_2^* d_{2R} + \omega\eta_3^* d_{3R}) \\ &+ h_2^d\bar{Q}_{2L}(\eta_1^* d_{1R} + \omega\eta_2^* d_{2R} + \omega^2\eta_3^* d_{3R}) \\ &+ h_3^d\bar{Q}_{3L}(\phi_1 d_{1R} + \phi_2 d_{2R} + \phi_3 d_{3R}) \\ &+ h_1^u\bar{Q}_{1L}(\phi_1^* u_{1R} + \omega^2\phi_2^* u_{2R} + \omega\phi_3^* u_{3R}) \\ &+ h_2^u\bar{Q}_{2L}(\phi_1^* u_{1R} + \omega\phi_2^* u_{2R} + \omega^2\phi_3^* u_{3R}) \\ &+ h_3^u\bar{Q}_{3L}(\eta_1 u_{1R} + \eta_2 u_{2R} + \eta_3 u_{3R}) + H.c, \end{aligned} \tag{21}$$

where a residual symmetry of lepton number $P_l \equiv (-1)^L$, called "lepton parity" [22, 24] has been introduced in order to suppress the mixing between ordinary quarks and exotic quarks. In this framework we assume that the lepton parity is an exact symmetry, not spontaneously broken. This means that due to the lepton parity conservation, the fields carrying lepton number ($L = \pm 1$) η_3 and χ_1 cannot develop VEV. Suppose that, under T_{13}, the VEVs of χ and η are $\langle\chi\rangle$ and $\langle\eta\rangle = (\langle\eta_1\rangle, \langle\eta_1\rangle, \langle\eta_1\rangle)$, respectively, where

$$\langle\chi\rangle = (0 \quad 0 \quad v_\chi)^T, \qquad \langle\eta_1\rangle = (u \quad 0 \quad 0)^T, \tag{22}$$

The quark mass Lagrangian are

$$
\begin{aligned}
-\mathcal{L}_q^{mass} &= f_3 v_\chi \bar{U}_L U_R + f_1 v_\chi \bar{D}_{1L} D_{1R} + f_2 v_\chi \bar{D}_{2L} D_{2R} \\
&+ h_1^d u \bar{d}_{1L} d_{1R} + \omega^2 h_1^d u \bar{d}_{1L} d_{2R} + \omega h_1^d u \bar{d}_{1L} d_{3R} \\
&+ h_2^d u \bar{d}_{2L} d_{1R} + \omega h_2^d u \bar{d}_{2L} d_{2R} + \omega^2 h_2^d u \bar{d}_{2L} d_{3R} \\
&+ h_3^d v \bar{d}_{3L} d_{1R} + h_3^d v \bar{d}_{3L} d_{2R} + h_3^d v \bar{d}_{3L} d_{3R} \\
&- h_1^u v \bar{u}_{1L} u_{1R} - \omega^2 h_1^u v \bar{u}_{1L} u_{2R} - \omega h_1^u v \bar{u}_{1L} u_{3R} \\
&- h_2^u v \bar{u}_{2L} u_{1R} - \omega h_2^u v \bar{u}_{2L} u_{2R} - \omega^2\ h_2^u v \bar{u}_{2L} u_{3R} \\
&+ h_3^u u \bar{u}_{3L} u_{1R} + h_3^u u \bar{u}_{3L} u_{2R} + h_3^u u \bar{u}_{3L} u_{3R} + H.c.
\end{aligned} \tag{23}
$$

The exotic quarks get masses

$$
m_U = f_3 v_\chi, \quad m_{D_{1,2}} = f_{1,2} v_\chi, \tag{24}
$$

and the mass Lagrangian of the ordinary quarks reads:

$$
\begin{aligned}
-\mathcal{L}_q^{mass} &= h_1^d u \bar{d}_{1L} d_{1R} + \omega^2 h_1^d u \bar{d}_{1L} d_{2R} + \omega h_1^d u \bar{d}_{1L} d_{3R} \\
&+ h_2^d u \bar{d}_{2L} d_{1R} + \omega h_2^d u \bar{d}_{2L} d_{2R} + \omega^2 h_2^d u \bar{d}_{2L} d_{3R} \\
&+ h_3^d v \bar{d}_{3L} d_{1R} + h_3^d v \bar{d}_{3L} d_{2R} + h_3^d v \bar{d}_{3L} d_{3R} \\
&- h_1^u v \bar{u}_{1L} u_{1R} - \omega^2 h_1^u v \bar{u}_{1L} u_{2R} - \omega h_1^u v \bar{u}_{1L} u_{3R} \\
&- h_2^u v \bar{u}_{2L} u_{1R} - \omega h_2^u v \bar{u}_{2L} u_{2R} - \omega^2\ h_2^u v \bar{u}_{2L} u_{3R} \\
&+ h_3^u u \bar{u}_{3L} u_{1R} + h_3^u u \bar{u}_{3L} u_{2R} + h_3^u u \bar{u}_{3L} u_{3R} + H.c. \\
&= (\bar{u}_{1L} \quad \bar{u}_{2L} \quad \bar{u}_{3L}) M_u (u_{1R} \quad u_{2R} \quad u_{3R})^T \\
&+ (\bar{d}_{1L} \quad \bar{d}_{2L} \quad \bar{d}_{3L}) M_d (d_{1R} \quad d_{2R} \quad d_{3R})^T + H.c.
\end{aligned} \tag{25}
$$

From (25), the mass matrices for the ordinary up-quarks and down-quarks are, respectively, obtained as follows:

$$
M_u = \begin{pmatrix} -h_1^u v & -h_1^u \omega^2 v & -h_1^u \omega v \\ -h_2^u v & -h_2^u \omega v & -h_2^u \omega^2 v \\ h_3^u u & h_3^u u & h_3^u u \end{pmatrix}, \quad M_d = \begin{pmatrix} h_1^d u & h_1^d \omega^2 u & h_1^d \omega u \\ h_2^d u & h_2^d \omega u & h_2^d \omega^2 u \\ h_3^d v & h_3^d v & h_3^d v \end{pmatrix}. \tag{26}
$$

The matrices M_u, M_d in (26) are, respectively, diagonalized as

$$
\begin{aligned}
V_L^{u+} M_u V_R^u &= \begin{pmatrix} -\sqrt{3}h_1^u v & 0 & 0 \\ 0 & -\sqrt{3}h_2^u v & 0 \\ 0 & 0 & \sqrt{3}h_3^u u \end{pmatrix} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \\
V_L^{d+} M_d V_R^d &= \begin{pmatrix} \sqrt{3}h_1^d u & 0 & 0 \\ 0 & \sqrt{3}h_2^d u & 0 \\ 0 & 0 & \sqrt{3}h_3^d v \end{pmatrix} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}. \qquad (27)
\end{aligned}
$$

where

$$
V_R^u = V_R^d = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{pmatrix}. \qquad (28)
$$

The unitary matrices, which couple the left-handed up- and down -quarks to those in the mass bases, are $U_L^u = 1$ and $U_L^d = 1$, respectively. Therefore we get the CKM matrix at the tree level:

$$
U_{\mathrm{CKM}} = U_L^{d\dagger} U_L^u = 1. \qquad (29)
$$

This is a good approximation for the realistic quark mixing matrix, which implies that the mixings among the quarks are dynamically small. The current mass values for the quarks are given by [32]

$$
\begin{aligned}
m_u &= (1.8 \div 3.0)\ \mathrm{MeV}, \quad m_d = (4.5 \div 5.3)\ \mathrm{MeV}, \quad m_c = (1.25 \div 1.30)\ \mathrm{GeV}, \qquad (30) \\
m_s &= (90.0 \div 100.0)\ \mathrm{MeV}, \ m_t = (171.99 \div 174.43)\ \mathrm{GeV}, \ m_b = (4.15 \div 4.21)\ \mathrm{GeV}.
\end{aligned}
$$

It is obvious that if $|u| \sim |v|$, the quark Yukawa couplings can be evaluated from (27) and (30) which is the same as in [158]. We note that with the alignments of χ, η as in (22), $T_{13} \to Z_3$, i.e, in quark sector, there remains an residual symmetry Z_3. If Z_3 is broken in to {Identity} by another $SU(3)_L$ multiplet, there will be a contribution for quark sector. A detailed study on this problem has been studied in [159], so we will not discuss it further.

V. NEUTRINO MASS AND MIXING

The neutrino masses arise from the couplings of $\bar{\psi}^c_L \psi_L$ to scalars, where $\bar{\psi}^c_L \psi_L$ transforms as $3^* \oplus 6$ under $\mathrm{SU}(3)_L$ and $\underline{\bar{3}}_1 \oplus \underline{\bar{3}}_1 \oplus \underline{3}_2$ under T_{13}. For the known scalar triplets (ϕ, χ, η),

the available interactions are only $(\bar{\psi}^c_{iL}\psi_{iL})\phi$ but explicitly suppressed because of the $\mathcal{L}$-symmetry. We will therefore propose new $\mathrm{SU}(3)_L$ antisextets, lying in either $\underline{3}_1$ or $\underline{\bar{3}}_2$ under T_{13}, interact with $\bar{\psi}^c_L\psi_L$ to produce masses for the neutrino. The antisextet transforms as $s = (s_1, s_2, s_3) \sim \underline{\bar{3}}_2$ under T_{13}, with

$$s_i \ = \begin{pmatrix} s^0_{11} & s^+_{12} & s^0_{13} \\ s^+_{12} & s^{++}_{22} & s^+_{23} \\ s^0_{13} & s^+_{23} & s^0_{33} \end{pmatrix}_i \sim [6^*, 2/3, -4/3], (i = 1, 2, 3) \tag{31}$$

where the numbered subscripts on the component scalars are the $\mathrm{SU}(3)_L$ indices, whereas $i = 1, 2, 3$ is that of T_{13}.

The VEV of s is set as $(\langle s_1\rangle, \langle s_2, \langle s_3\rangle)$ under T_{13}, in which

$$\langle s_i\rangle = \begin{pmatrix} \lambda_i & 0 & v_i \\ 0 & 0 & 0 \\ v_i & 0 & \Lambda_i \end{pmatrix}. \tag{32}$$

Following the potential minimization conditions, we have the followings alignments:

(1) The first alignment: $\langle s_1\rangle \neq \langle s_2\rangle \neq \langle s_3\rangle$ then T_{13} is completely broken.

(2) The second alignment: $0 \neq \langle s_1\rangle \neq \langle s_2\rangle = \langle s_3\rangle \neq 0$ or $0 \neq \langle s_1\rangle = \langle s_3\rangle \neq \langle s_2\rangle \neq 0$ or $0 \neq \langle s_1\rangle = \langle s_2\rangle \neq \langle s_3\rangle \neq 0$ then T_{13} is completely broken.

(3) The third alignment: $\langle s_1\rangle = \langle s_2\rangle = \langle s_3\rangle \neq 0$ then T_{13} is broken into Z_3 that consists of the elements $\{e, b, b^2\}$.

(4) The fourth alignment: $0 = \langle s_1\rangle \neq \langle s_2\rangle \neq \langle s_3\rangle \neq 0$ or $0 = \langle s_2\rangle \neq \langle s_1\rangle \neq \langle s_3\rangle \neq 0$ or $0 \neq \langle s_1\rangle \neq \langle s_2\rangle \neq \langle s_3\rangle = 0$ then T_{13} is completely broken.

(5) The fifth alignment: $0 = \langle s_1\rangle \neq \langle s_2\rangle = \langle s_3\rangle \neq 0$ or $0 = \langle s_2\rangle \neq \langle s_1\rangle = \langle s_3\rangle \neq 0$ or $0 \neq \langle s_1\rangle = \langle s_2\rangle \neq \langle s_3\rangle = 0$ then T_{13} is completely broken.

(6) The sixth alignment: $0 = \langle s_1\rangle = \langle s_2\rangle \neq \langle s_3\rangle$ or $0 = \langle s_1\rangle = \langle s_3\rangle \neq \langle s_2\rangle$ or $0 \neq \langle s_1\rangle \neq \langle s_2\rangle = \langle s_3\rangle = 0$ then T_{13} is completely broken.

In this work we impose both the breakings $T_{13} \to Z_3$ and $Z_3 \to \{\text{identity}\}$[2] (instead of $T_{13} \to \{\text{identity}\}$) must be taken place in neutrino sector. The $T_{13} \to Z_3$ is achieved with

[2] It means Z_3 is completely broken.

the alignment

$$\langle s_1\rangle = \langle s_1\rangle = \langle s_1\rangle = \langle s\rangle \neq 0, \tag{33}$$

where

$$\langle s\rangle \ = \begin{pmatrix} \lambda_s & 0 & v_s \\ 0 & 0 & 0 \\ v_s & 0 & \Lambda_s \end{pmatrix}, \tag{34}$$

or

$$\lambda_1 \ = \ \lambda_2 = \lambda_3 = \lambda_s, \ \ v_1 = v_2 = v_3 = v_s, \ \ \Lambda_1 = \Lambda_2 = \Lambda_3 = \Lambda_s. \tag{35}$$

The direction of the breaking $Z_3 \to \{\text{identity}\}$ (instead of $T_{13} \to \{\text{identity}\}$) is achieved in the case $\langle s\rangle = (\langle s\rangle, 0, 0)$ under T_{13}. To achieve the second direction of the breakings $Z_3 \to \{\text{Identity}\}$, we additionally introduce either another scalar $SU(3)_L$ anti-sextet or $SU(3)_L$ triplet which both lies in $\underline{3}_1$ under T_{13}. We can therefore understand the misalignment of the VEVs in neutrino sector as follows. The T_{13} is broken via two stages, the first stage is $T_{13} \to Z_3$ and the second stage is $T_{13} \to \{\text{identity}\}$. The first stage is achieved by a $\mathrm{SU}(3)_L$ anti-sextet s with the alignment as in (33). The second stage can be achieved within each case below:

1. A new $\mathrm{SU}(3)_L$ anti-sextet σ lies in $\underline{3}_1$ under T_{13} ,

$$\sigma = \begin{pmatrix} \sigma^0_{11} & \sigma^+_{12} & \sigma^0_{13} \\ \sigma^+_{12} & \sigma^{++}_{22} & \sigma^+_{23} \\ \sigma^0_{13} & \sigma^+_{23} & \sigma^0_{33} \end{pmatrix}_i \sim [6^*, 2/3, -4/3, \underline{3}_1], \tag{36}$$

with VEVs is given by $\langle\sigma\rangle = (\langle\sigma_1\rangle, 0, 0)$ under T_{13}, where

$$\langle\sigma_1\rangle = \begin{pmatrix} \lambda_\sigma & 0 & v_\sigma \\ 0 & 0 & 0 \\ v_\sigma & 0 & \Lambda_\sigma \end{pmatrix}, \ \langle\sigma_2\rangle = \langle\sigma_3\rangle = 0. \tag{37}$$

2. Another antisextet s' lies in $\underline{\bar{3}}_2$ under T_{13},

$$s'_i = \begin{pmatrix} s'^0_{11} & s'^+_{12} & s'^0_{13} \\ s'^+_{12} & s'^{++}_{22} & s'^+_{23} \\ s'^0_{13} & s'^+_{23} & s'^0_{33} \end{pmatrix}_i \sim [6^*, 2/3, -4/3, \underline{\bar{3}}_2], \tag{38}$$

with VEVs is given by $\langle s'\rangle = (0, \langle s'_2\rangle, 0)$ under T_{13}, where

$$\langle s'_2\rangle = \begin{pmatrix} \lambda'_s & 0 & v'_s \\ 0 & 0 & 0 \\ v'_s & 0 & \Lambda'_s \end{pmatrix}. \tag{39}$$

In calculation, combining both cases we have the Yukawa interactions:

$$\begin{aligned} -\mathcal{L}_\nu &= \frac{1}{2}x(\bar{\psi}^c_L\psi_L)_{\underline{3}_2}s + \frac{1}{2}y_1(\bar{\psi}^c_L\psi_L)_{\underline{\bar{3}}_1}\sigma \\ &+ \frac{1}{2}y_2(\bar{\psi}^c_L\psi_L)_{\underline{\bar{3}}_1}\sigma + \frac{1}{2}z(\bar{\psi}^c_L\psi_L)_{\underline{3}_2}s' + H.c. \\ &= \frac{1}{2}x\left[(\bar{\psi}^c_{1L}\psi_{1L})s_1 + (\bar{\psi}^c_{2L}\psi_{2L})s_2 + (\bar{\psi}^c_{3L}\psi_{3L})s_3\right] \\ &+ \frac{1}{2}y_1\left[(\bar{\psi}^c_{2L}\psi_{3L})\sigma_1 + (\bar{\psi}^c_{3L}\psi_{1L})\sigma_2 + (\bar{\psi}^c_{1L}\psi_{2L})\sigma_3\right] \\ &+ \frac{1}{2}y_2\left[(\bar{\psi}^c_{3L}\psi_{2L})\sigma_1 + (\bar{\psi}^c_{1L}\psi_{3L})\sigma_2 + (\bar{\psi}^c_{2L}\psi_{1L})\sigma_3\right] \\ &+ \frac{1}{2}z\left[(\bar{\psi}^c_{1L}\psi_{1L})s'_1 + (\bar{\psi}^c_{2L}\psi_{2L})s'_2 + (\bar{\psi}^c_{3L}\psi_{3L})s'_3\right] + H.c. \end{aligned} \tag{40}$$

Institute (34), (37) and (39) into (40) we obtain the mass Lagrangian for the neutrinos:

$$\begin{aligned} -\mathcal{L}^{mass}_\nu &= \frac{1}{2}x\left(\lambda_s\bar{\nu}^c_{1L}\nu_{1L} + v_s\bar{N}_{1R}\nu_{1L} + v_s\bar{\nu}^c_{1L}N^c_{1R} + \Lambda_s\bar{N}_{1R}N^c_{1R}\right. \\ &+ \lambda_s\bar{\nu}^c_{2L}\nu_{2L} + v_s\bar{N}_{2R}\nu_{2L} + v_s\bar{\nu}^c_{2L}N^c_{2R} + \Lambda_s\bar{N}_{2R}N^c_{2R} \\ &+ \left.\lambda_s\bar{\nu}^c_{3L}\nu_{3L} + v_s\bar{N}_{3R}\nu_{3L} + v_s\bar{\nu}^c_{3L}N^c_{3R} + \Lambda_s\bar{N}_{3R}N^c_{3R}\right) \\ &+ \frac{1}{2}y_1\left(\lambda_\sigma\bar{\nu}^c_{2L}\nu_{3L} + v_\sigma\bar{N}_{2R}\nu_{3L} + v_\sigma\bar{\nu}^c_{2L}N^c_{3R} + \Lambda_\sigma\bar{N}_{2R}N^c_{3R}\right) \\ &+ \frac{1}{2}y_2\left(\lambda_\sigma\bar{\nu}^c_{3L}\nu_{2L} + v_\sigma\bar{N}_{3R}\nu_{2L} + v_\sigma\bar{\nu}^c_{3L}N^c_{2R} + \Lambda_\sigma\bar{N}_{3R}N^c_{2R}\right) \\ &+ \frac{1}{2}z\left(\lambda'_s\bar{\nu}^c_{2L}\nu_{2L} + v'_s\bar{N}_{2R}\nu_{2L} + v'_s\bar{\nu}^c_{2L}N^c_{2R} + \Lambda'_s\bar{N}_{2R}N^c_{2R}\right) + H.c. \end{aligned} \tag{41}$$

$-\mathcal{L}^{mass}_\nu$ in (41) can be rewriten in the matrix form:

$$-\mathcal{L}^{\mathrm{mass}}_\nu = \frac{1}{2}\bar{\chi}^c_L M_\nu \chi_L + H.c., \tag{42}$$

where

$$\begin{aligned} \chi_L &\equiv \begin{pmatrix} \nu_L \\ N^c_R \end{pmatrix}, \quad M_\nu \equiv \begin{pmatrix} M_L & M^T_D \\ M_D & M_R \end{pmatrix}, \\ \nu_L &= (\nu_{1L}, \nu_{2L}, \nu_{3L})^T, \quad N_R = (N_{1R}, N_{2R}, N_{3R})^T, \end{aligned} \tag{43}$$

and

$$M_{L,R,D} = \begin{pmatrix} a_{L,R,D} & 0 & 0 \\ 0 & a_{L,R,D} + c_{L,R,D} & b_{L,R,D} \\ 0 & b_{L,R,D} & a_{L,R,D} \end{pmatrix}, \tag{44}$$

with

$$\begin{aligned} a_L &= \lambda_s x, \qquad a_D = v_s x, \qquad a_R = \Lambda_s x, \\ c_L &= \lambda'_s z, \qquad c_D = v'_s z, \qquad c_R = \Lambda'_s z, \\ b_L &= \frac{\lambda_\sigma}{2}(y_1 + y_2), \quad b_D = \frac{v_\sigma}{2}(y_1 + y_2), \quad b_R = \frac{\Lambda_\sigma}{2}(y_1 + y_2). \end{aligned} \tag{45}$$

Three observed neutrinos gain masses via a combination of type I and type II seesaw mechanisms derived from (42) as

$$M_{\text{eff}} = M_L - M_D^T M_R^{-1} M_D = \begin{pmatrix} A & 0 & 0 \\ 0 & B_1 & C \\ 0 & C & B_2 \end{pmatrix}, \tag{46}$$

where

$$\begin{aligned} A &= a_L - \frac{a_D^2}{a_R}, \\ B_1 &= a_L - \frac{[b_D^2 + (a_D + c_D)^2 - c_L(a_R + c_R)]a_R}{a_R^2 - b_R^2 + a_R c_R} + \frac{b_R[2b_D(a_D + c_D) - b_R c_L] - b_D^2 c_R}{a_R^2 - b_R^2 + a_R c_R}, \\ B_2 &= a_L - \frac{(a_D^2 + b_D^2)a_R}{a_R^2 - b_R^2 + a_R c_R} - \frac{(2b_D b_R - a_D c_R)a_D}{a_R^2 - b_R^2 + a_R c_R}, \\ C &= b_L - \frac{(b_D^2 + a_D^2 + a_D c_R)b_R}{a_R^2 - b_R^2 + a_R c_R} + \frac{(2a_D a_R + a_D c_R + a_R c_D)b_D}{a_R^2 - b_R^2 + a_R c_R}. \end{aligned} \tag{47}$$

The matrix M_{eff} in (46) has three exact eigenvalues given by

$$\begin{aligned} \lambda_1 &= A, \\ \lambda_{2,3} &= \frac{1}{2}\left(B_1 + B_2 \mp \sqrt{(B_1 - B_2)^2 + 4C^2}\right), \end{aligned} \tag{48}$$

and the corresponding eigenstates are

$$\varphi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 0 \\ \frac{K}{\sqrt{K^2+1}} \\ \frac{1}{\sqrt{K^2+1}} \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{K^2+1}} \\ -\frac{K}{\sqrt{K^2+1}} \end{pmatrix}, \tag{49}$$

where

$$K = \frac{B_1 - B_2 - \sqrt{(B_1 - B_2)^2 + 4C^2}}{2C}. \tag{50}$$

Until now the values of neutrino masses (or the absolute neutrino masses) as well as the mass ordering of neutrinos are unknown. The neutrino mass spectrum can be the normal hierarchy ($|m_1| \simeq |m_2| < |m_3|$), the inverted hierarchy ($|m_3| < |m_1| \simeq |m_2|$) or nearly degenerate ($|m_1| \simeq |m_2| \simeq |m_3|$). An upper bound on the absolute value of neutrino mass was found from the analysis of the cosmological data [165]

$$m_i \leq 0.6\,\mathrm{eV}, \tag{51}$$

while the upper limit on the sum of neutrino masses given in [166]

$$\sum_{i=1}^{3} m_i < 0.23\,\mathrm{eV} \tag{52}$$

In the case of 3-neutrino mixing, the two possible signs of Δm^2_{23} corresponding to two types of neutrino mass spectrum can be provided as follows:

- Normal hierarchy (NH): $|m_1| \simeq |m_2| < |m_3|,\ \Delta m^2_{32} = m^2_3 - m^2_2 > 0.$
- Inverted hierarchy (IH): $|m_3| < |m_1| \simeq |m_2|,\ \Delta m^2_{32} = m^2_3 - m^2_2 < 0.$

As will be discussed below, the model under consideration can provide both normal and inverted mass hierarchy.

A. Normal hierarchy ($\Delta m^2_{32} > 0$)

In this case, three neutrino masses are

$$m_1 = |\lambda_3|\,,\ m_2 = |\lambda_1|\,,\ m_3 = |\lambda_2|\,, \tag{53}$$

with $\lambda_i\ \ (i = 1, 2, 3)$ is defined in (49), and the corresponding eigenstates put in the neutrino mixing matrix:

$$U_\nu = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{K^2+1}} & 0 & \frac{K}{\sqrt{K^2+1}} \\ -\frac{K}{\sqrt{K^2+1}} & 0 & \frac{1}{\sqrt{K^2+1}} \end{pmatrix} .P, \tag{54}$$

where $P = \mathrm{diag}(1,\ 1,\ i)$. Combining (16) and (54) we get the lepton mixing matrix:

$$U_L^\dagger U_\nu = \frac{1}{\sqrt{3}}\begin{pmatrix} \frac{1-K}{\sqrt{K^2+1}} & 1 & \frac{1+K}{\sqrt{K^2+1}} \\ \frac{\omega(1-K\omega)}{\sqrt{K^2+1}} & 1 & \frac{\omega(\omega+K)}{\sqrt{K^2+1}} \\ \frac{\omega(\omega-K)}{\sqrt{K^2+1}} & 1 & \frac{\omega(K\omega+1)}{\sqrt{K^2+1}} \end{pmatrix}.P. \tag{55}$$

We note that in the case $K = -1$, $U_L^\dagger U_\nu \equiv U_{\mathrm{HPS}}$ which is given in (2).

The value of the Jarlskog invariant J_{CP}, which gives a convention-independent measure of CP violation, is defined from (55) as

$$J_{CP} = \mathrm{Im}[U_{21}U_{31}^* U_{22}^* U_{32}] = \frac{\sqrt{3}(K^2-1)}{18(K^2+1)}. \tag{56}$$

Combining (56) with the data in [32] for Normal Hierarchy,

$$J_{CP} = -0.032 \tag{57}$$

we find the corresponding values of K,

$$K \ = \ -0.708 \quad (\mathrm{NH}), \tag{58}$$

and the lepton mixing matrices are obtained as

$$U_{lep} \ = \begin{pmatrix} 0.805 & \frac{1}{\sqrt{3}} & 0.138 \\ -0.402+0.119i & \frac{1}{\sqrt{3}} & 0.697-0.069i \\ -0.402-0.119i & \frac{1}{\sqrt{3}} & -0.697-0.069i \end{pmatrix} \times P, \tag{59}$$

or

$$|U_{lep}| \ = \begin{pmatrix} 0.805 & 0.577 & 0.138 \\ 0.420 & 0.577 & 0.700 \\ 0.420 & 0.577 & 0.700 \end{pmatrix}, \tag{60}$$

In the standard parametrization, the lepton mixing matrix (U_{PMNS}) can be parametrized as

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}.\mathcal{P}, \tag{61}$$

where $\mathcal{P} = \mathrm{diag}(1, e^{i\alpha}, e^{i\beta})$, and $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$ with θ_{12}, θ_{23} and θ_{13} being the solar, atmospheric and reactor angles, respectively. $\delta = [0, 2\pi]$ is the Dirac CP violation

phase while α and β are two Majorana CP violation phases. Using the parametrization in Eq. (61) we get

$$J_{CP} = \frac{1}{8}\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{23}\sin 2\theta_{13}\sin\delta. \quad (62)$$

Combining (62) and (57) with the data given in (4) yields $\sin\delta_{CP} = -0.924$, i.e, $\delta_{CP} = -67.5^\circ$ or $\delta_{CP} = 292.5^\circ$.

From Eqs. (50) and (58) we get

$$B_1 = B_2 + 0.704429C. \quad (63)$$

In normal case, i.e, $\Delta m^2_{32} = m^2_3 - m^2_2 > 0$, for the remaining constraints, taking the central values from the data in [32] as shown in (4): $\Delta m^2_{21} = 7.53\times10^{-5}$ eV^2 and $\Delta m^2_{32} = 2.44\times10^{-3}$ eV^2, with $m_{1,2,3}$ given in Eqs. (48), we get a solution [3] (in [eV])

$$\begin{aligned} B_2 &= -0.5\sqrt{4A^2 - 0.0003} - 1.41243C, \\ C &= 0.47160\left(\sqrt{A^2 + 2.44\times10^{-3}} - \sqrt{A^2 - 7.53\times10^{-5}}\right). \end{aligned} \quad (64)$$

With $B_{1,2}$ and C in Eqs. (63) and (64), $m_{1,2,3}$ depends only on one parameter $A \equiv m_2$, so we will consider $m_{1,3}$ as functions of A. However, to have an explicit hierarchy on neutrino masses, in the following figures, m_2 should be included. By using the upper bound on the absolute value of neutrino mass in (51) we can restrict the values of A: $A \leq 0.6$ eV. However, in this case, $A \in (0.0087, 0.05)$ eV or $A \in (-0.05, -0.0087)$ eV are good regions of A that can reach the realistic neutrino mass hierarchy.

In Fig. 1, we have plotted the absolute value $|m_{1,2,3}|$ as functions of A with $A \in (0.0087, 0.05)$ eV and $A \in (-0.05, -0.0087)$ eV, respectively. This figure shows that there exist allowed regions for values A where either normal or quasi-degenerate neutrino masses spectrum is achieved. The quasi-degenerate mass hierarchy is obtained when A lies in a region[4] $[0.05\,\text{eV}, +\infty)$. The normal mass hierarchy will be obtained if A takes the values around $(0.0087, 0.05)$ eV or $(-0.05, -0.0087)$ eV. The sum of neutrino masses $\sum = \sum_{i=1}^{3}|m_i|$ with $A \in (0.0087, 0.05)$ eV is depicted in Fig. 2 which is consistent with the upper limit given in Eq.(52).

[3] In fact, this system of equations has four solutions, however, these equations differ only by the sign of $m_{1,2,3}$ that it is not appear in the neutrino oscillation experiments. So, here we only consider in detail the solution in (64)

[4] A increases but must be small enough because of the scale of $|m_{1,2,3}|$

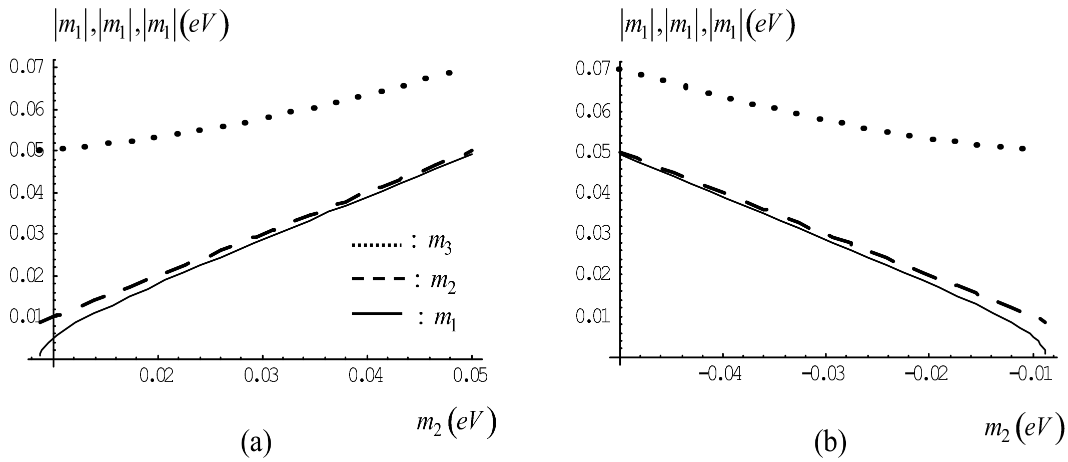

FIG. 1: $|m_{1,2,3}|$ as functions of A in the normal hierarchy with a) $A \in (0.0087, 0.05)\,\mathrm{eV}$ and b) $A \in (-0.05, -0.0087)\,\mathrm{eV}$.

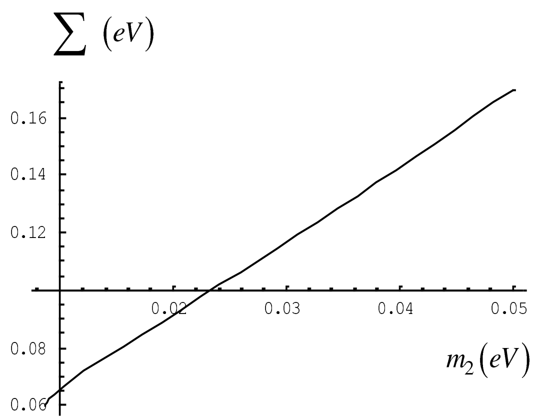

FIG. 2: $\sum$ as a function of A in the normal hierarchy.

From the expressions (50), (55) and (64), it is easily to obtain the effective masses governing neutrinoless double beta decay [168–172],

$$m_{ee}^{N} = \sum_{i=1}^{3} U_{ei}^{2}\,|m_i|\,, \quad m_{\beta}^{N} = \sqrt{\sum_{i=1}^{3} |U_{ei}|^2\, m_i^2} \tag{65}$$

which is plotted in Fig. 3 with $A \in (0.0087, 0.05)\,\mathrm{eV}$ in the case of $\Delta m_{32}^2 > 0$. We also note that in the normal spectrum, $|m_1| \approx |m_2| < |m_3|$, so m_1 given in (48) is the lightest neutrino mass, which is denoted as $m_1 \equiv m_{light}^{N}$.

To get explicit values of the model parameters, we set $A = 10^{-2}\,\mathrm{eV}$, which is safely small.

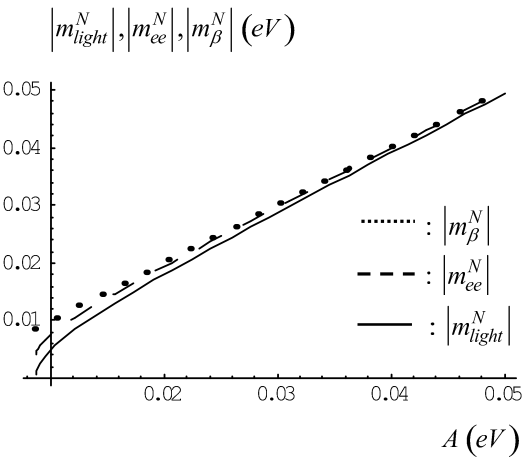

FIG. 3: $|m^N_{light}|$, $|m^N_{ee}|$ and $|m^N_\beta|$ as functions of A with $A \in (0.0087, 0.05)$ eV in the normal hierarchy.

Then the other physical neutrino masses are explicitly given as

$$|m_1| \simeq 4.97 \times 10^{-3}\,\mathrm{eV}, \quad |m_2| = 10^{-2}\,\mathrm{eV}, \quad |m_3| \simeq 5.04 \times 10^{-2}\,\mathrm{eV}. \tag{66}$$

It follows that

$$|m^N_{ee}| \simeq 7.51 \times 10^{-3}\,\mathrm{eV}, \quad |m^N_\beta| = 9.87 \times 10^{-3}\,\mathrm{eV}, \tag{67}$$

$$B_1 = -2.014 \times 10^{-2}\,\mathrm{eV}, \; B_2 = -3.523 \times 10^{-2}\,\mathrm{eV}, \; C = 2.142 \times 10^{-2}\,\mathrm{eV}. \tag{68}$$

This solution means a normal neutrino mass spectrum as mentioned above. Furthermore, by assuming that[5]

$$\lambda_s = \lambda'_s = \lambda_\sigma = 1\,\mathrm{eV}, \; v_s = v'_s = v_\sigma, \; \Lambda'_s = \Lambda_\sigma = -\Lambda_s, \Lambda_s = -v_s^2, \tag{69}$$

we obtain a solution

$$x = 5 \times 10^{-3}, \; y_1 = y_2 \simeq -1.05 \times 10^{-2}, \; z \simeq -7.46 \times 10^{-3}. \tag{70}$$

B. Inverted hierarchy ($\Delta m^2_{32} < 0$)

For inverted hierarchy, three neutrino masses are

$$m_1 = |\lambda_2|\,, \; m_2 = |\lambda_1|\,, \; m_3 = |\lambda_3|\,, \tag{71}$$

[5] The values of the parameters $\lambda_s, \lambda'_s, \lambda_\sigma, v_s, v'_s, v_\sigma, \Lambda_s, \Lambda'_s, \Lambda_\sigma$ have not been confirmed by by experiment, however, their hierarchies were given in [24, 167]. The parametres in Eqs. (69) and (70) is a set of the model parameters that can fit the experimenta data on neutrino given in (4)

with λ_i $(i=1,2,3)$ is defined in (49), and the corresponding lepton mixing matrix becomes

$$U_L^\dagger U_\nu = \frac{1}{\sqrt{3}}\begin{pmatrix} \frac{1+K}{\sqrt{K^2+1}} & 1 & \frac{1-K}{\sqrt{K^2+1}} \\ \frac{\omega(\omega+K)}{\sqrt{K^2+1}} & 1 & \frac{\omega(1-K\omega)}{\sqrt{K^2+1}} \\ \frac{\omega(K\omega+1)}{\sqrt{K^2+1}} & 1 & \frac{\omega(\omega-K)}{\sqrt{K^2+1}} \end{pmatrix}.P. \tag{72}$$

The value of the Jarlskog invariant J_{CP} is defined from (72) as

$$J_{CP} = \frac{\sqrt{3}(1-K^2)}{18(1+K^2)}. \tag{73}$$

In the inverted hierarchy [32], $J_{CP} = -0.029$, we get

$$K = 1.365, \tag{74}$$

$$B_1 = B_2 + 0.63213C, \tag{75}$$

and the lepton mixing matrices in (72) becomes

$$U^I_{lep} = \begin{pmatrix} 0.807 & \frac{1}{\sqrt{3}} & -0.125 \\ -0.403+0.108i & \frac{1}{\sqrt{3}} & 0.062+0.699i \\ -0.403-0.108i & \frac{1}{\sqrt{3}} & 0.062-0.699i \end{pmatrix} \times P. \tag{76}$$

In the case of the inverted spectrum, combining (62) with the data given in [32], $J_{CP} = -0.029$, we get $\sin\delta_{CP} = -0.837136$, i.e, $\delta_{CP} = -56.839^\circ$ or $\delta_{CP} = 303.161^\circ$.

Now, by taking the central values from the data in [32] as shown in (4): $\Delta m^2_{21} = 7.53\times 10^{-5}$ eV^2 and $\Delta m^2_{32} = -2.52\times 10^{-3}$ eV^2, with $m_{1,2,3}$ is given in (71) and $l_{1,2,3}$ in Eq. (49), we get a solution (in [eV]) as follows [6]:

$$\begin{aligned} B_2 &= 0.349313\sqrt{A^2-2.52\times 10^{-3}} - 0.650687\sqrt{A^2-7.53\times 10^{-5}}, \\ C &= 0.476753\left(\sqrt{A^2-7.53\times 10^{-5}} - \sqrt{A^2-2.52\times 10^{-3}}\right). \end{aligned} \tag{77}$$

In this case, $A \in (0.05, 0.1)\,$eV or $A \in (-0.01, -0.05)\,$eV are good regions of A that can reach the realistic neutrino mass hierarchy. Eqs. (77) , (71) and (49) show that $m_{1,2,3}$ only depends on one parameter $A \equiv m_2$, so we will consider $m_{1,3}$ as functions of A. However, to have an explicit hierarchy on neutrino masses, in the following figures, m_2 should be

[6] Similarly to in the normal case, there are four solutions in the inverted hierarchy. Here we only consider in detail the solution in (77)

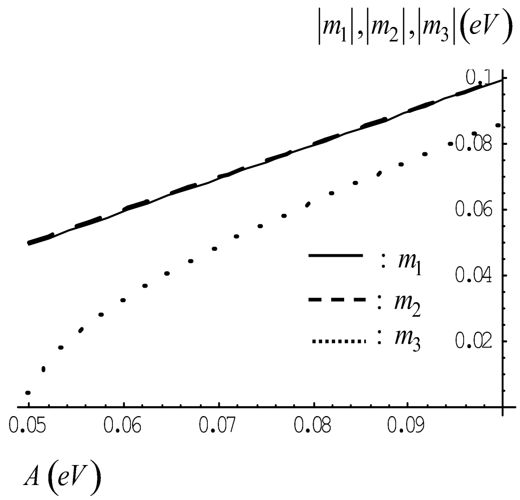

FIG. 4: $|m_{1,2,3}|$ as functions of A with $A \in (0.05, 0.1)\,\mathrm{eV}$ in the inverted hierarchy.

included. In Fig. 4, we have plotted the absolute value $|m_{1,2,3}|$ as functions of A with $A \in (0.0502, 0.07)\,\mathrm{eV}$ in the inverted spectrum.

Fig. 4 shows that there exist allowed regions for values of A where either inverted or quasi-degenerate neutrino masses spectrum is achieved. The quasi-degenerate mass hierarchy is obtained if $|A| \in [0.1\,\mathrm{eV}, +\infty)$ [7]. The inverted mass hierarchy will be obtained if A takes the values around $(0.05, 0.1)\,\mathrm{eV}$ or $(-0.1, -0.05)\,\mathrm{eV}$. The sum $\sum^I = \sum_{i=1}^3 |m_i|$ with $A \in (0.050, 0.1)\,\mathrm{eV}$ is plotted in the Fig. 5.

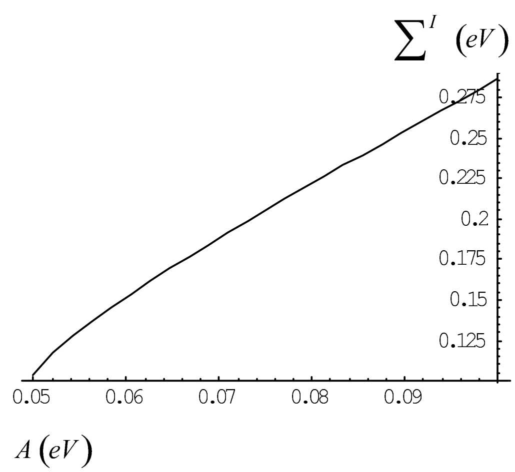

FIG. 5: The sum $\sum^I = \sum_{i=1}^3 \mid m_i \mid$ as a function of A with $A \in (0.05, 0.1)\,\mathrm{eV}$ in the case of $\Delta m_{32}^2 < 0$.

[7] A increases but must be small enough because of the scale of $|m_{1,2,3}|$

In the inverted spectrum, $|m_3| < |m_1| \approx |m_2|$, so $m_3 \equiv m^I_{light}$ given in (71) is the lightest neutrino mass. In Fig. 6, we have plotted the values of $|m^I_{ee}|$, $|m^I_\beta|$ and $|m^I_{light}|$ as functions of A with $A \in (0.05, 0.1)$ eV in the inverted spectrum.

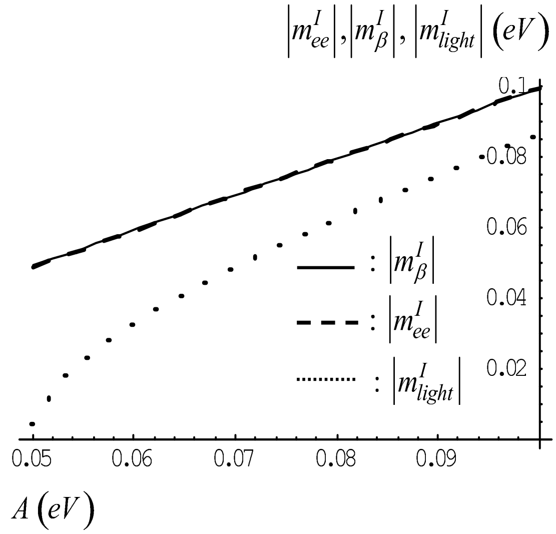

FIG. 6: $|m^I_{ee}|$, $|m^I_\beta|$ and $|m^I_{light}|$ as functions of A with $A \in (0.05, 0.1)$ eV in the case of $\Delta m^2_{32} < 0$.

To get explicit values of the model parameters, we set $A = .075$ eV, which is safely small. Then the other physical neutrino masses are explicitly given as

$$|m_1| = 0.0744963\,\mathrm{eV}, \quad |m_2| = 0.075\,\mathrm{eV}, \quad |m_3| = 0.0557225\,\mathrm{eV}. \tag{78}$$

It follows that $B_1 = 0.0102354$ eV, $B_2 = -0.0290092$ eV and $C = 0.0620822$ eV. Furthermore, by assuming that

$$\lambda_s = \lambda'_s = \lambda_\sigma = 1\,\mathrm{eV},\ v_s = v'_s = v_\sigma,\ \Lambda_s = a v_s^2, \Lambda'_s = v'^2_s, \Lambda_\sigma = v_\sigma^2,\ y_1 = y_2 = y, \tag{79}$$

we obtain a solution as follows

$$x = -0.128008, \quad y = -0.135227, \quad z = -0.0854825, \quad a = 0.630556. \tag{80}$$

VI. CONCLUSIONS

In this paper, we have studied a 3-3-1 model based on non-Abelian discrete symmetry group T_{13} responsible for lepton mixing with non-zero θ_{13} and CP violation phase. The

neutrinos get small masses and mixing with CP violation phase from $SU(3)_L$ antisextets which are all in triplets under T_{13}. If both breakings $T_{13} \rightarrow Z_3$ and $Z_3 \rightarrow \{\text{Identity}\}$ are taken palce in neutrino sector and T_{13} is broken into Z_3 in lepton sector, the realistic neutrino mixing form has been obtained as a natural consequence of P_l and T_{13} symmetries. The model predicts the lepton mixing with non-zero θ_{13}, and also gives a remarkable prediction of Dirac CP violation $\delta_{CP} = 292.5^\circ$ in the normal spectrum, and $\delta_{CP} = 303.161^\circ$ in the inverted spectrum which is still missing in the neutrino mixing matrix. There exist some regions of model parameters that can fit the experimental data in 2014 on neutrino masses and mixing without perturbation.

Acknowledgments

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Appendix A: T_{13} group theory

The T_{13} group is a subgroup of $SU(3)$, and known as the minimal non-Abelian discrete group having two complex triplets as the irreducible representations. We denote the generators of Z_{13} and Z_3 by a and b, respectively. They satisfy

$$a^{13} = 1, \quad ab = ba^9, \quad b^3 = 1. \tag{A1}$$

All of T_{13} elements are written as

$$g = b^m a^n, \tag{A2}$$

with $m = 0, 1, 2$ and $n = 0, \cdots, 12$.

The generators, a and b, are represented e.g. as

$$a = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho^3 & 0 \\ 0 & 0 & \rho^9 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{A3}$$

	n	h	$\chi_{\mathbf{1_0}}$	$\chi_{\mathbf{1_1}}$	$\chi_{\mathbf{1_2}}$	$\chi_{\mathbf{3_1}}$	$\chi_{\bar{\mathbf{3}}_\mathbf{1}}$	$\chi_{\mathbf{3_2}}$	$\chi_{\bar{\mathbf{3}}_\mathbf{2}}$
$C_1^{(0)}$	1	1	1	1	1	3	3	3	3
$C_{13}^{(1)}$	13	3	1	ω	ω^2	0	0	0	0
$C_{13}^{(2)}$	13	3	1	ω^2	ω	0	0	0	0
C_{3_1}	3	13	1	1	1	ξ_1	$\bar{\xi}_1$	ξ_2	$\bar{\xi}_2$
$C_{\bar{3}_1}$	3	13	1	1	1	$\bar{\xi}_1$	ξ_1	$\bar{\xi}_2$	ξ_2
C_{3_2}	3	13	1	1	1	ξ_2	$\bar{\xi}_2$	ξ_1	$\bar{\xi}_1$
$C_{\bar{3}_2}$	3	13	1	1	1	$\bar{\xi}_2$	ξ_2	$\bar{\xi}_1$	ξ_1

TABLE I: Characters of T_{13} where $\bar{\xi}_i$ $(i=1,2)$ is defined as the complex conjugate of ξ_i.

where $\rho = e^{2i\pi/13}$. These elements are classified into seven conjugacy classes,

$$\begin{array}{llr}
C_1 : & \{e\}, & h=1, \\
C_{13}^{(1)} : & \{b \ , \ ba \ , \ ba^2 \ , \ \ \ldots \ \ , ba^{10} \ , \ ba^{11} \ , \ ba^{12}\}, & h=3, \\
C_{13}^{(2)} : & \{b^2 \ , \ b^2a \ , \ b^2a^2 \ , \ \ \ldots \ \ , b^2a^{10} \ , \ b^2a^{11} \ , \ b^2a^{12}\}, & h=3, \\
C_{3_1} : & \{a \ , \ a^3 \ , \ a^9\}, & h=13, \\
C_{\bar{3}_1} : & \{a^4 \ , \ a^{10} \ , \ a^{12}\}, & h=13, \\
C_{3_2} : & \{a^2 \ , \ a^5 \ , \ a^6\}, & h=13, \\
C_{\bar{3}_2} : & \{a^7 \ , \ a^8 \ , \ a^{11}\}, & h=13.
\end{array} \tag{A4}$$

The T_{13} group has three singlets $\mathbf{1}_k$ with $k=0,\ 1,\ 2$ and two complex triplets $\mathbf{3_1}$ and $\mathbf{3_2}$ as irreducible representations. The characters are shown in Table I, where $\xi_1 \equiv \rho+\rho^3+\rho^9$, $\xi_2 \equiv \rho^2+\rho^5+\rho^6$, and $\omega \equiv e^{2i\pi/3}$.

Now, let us put $\underline{3}(1,2,3)$ which means some $\underline{3}$ multiplet such as $x=(x_1,x_2,x_3) \sim \underline{3}$ or $y=(y_1,y_2,y_3) \sim \underline{3}$ and so on, and similarly for the other representations. Moreover, the numbered multiplets such as $(...,ij,...)$ mean $(...,x_iy_j,...)$ where x_i and y_j are the multiplet components of different representations x and y, respectively. In the following the components of representations in l.h.s will be omitted and should be understood, but they always exist in order in the components of decompositions in r.h.s. All the group multiplication

rules of T_{13} as given below. The tensor products between triplets are obtained as

$$\begin{aligned}
\underline{3}_1 \otimes \underline{3}_1 &= \underline{\bar{3}}_1(23,31,12) \oplus \underline{\bar{3}}_1(32,13,21) \oplus \underline{3}_2(11,22,33),\\
\underline{\bar{3}}_1 \otimes \underline{\bar{3}}_1 &= \underline{3}_1(23,31,12) \oplus \underline{3}_1(32,13,21) \oplus \underline{\bar{3}}_2(11,22,33),\\
\underline{\bar{3}}_1 \otimes \underline{3}_1 &= \underline{3}_1 \otimes \underline{\bar{3}}_1 = 1_0(11+22+33) \oplus 1_1(11+\omega 22+\omega^2 33)\\
&\oplus\ 1_2(11+\omega^2 22+\omega 33) \oplus \underline{3}_2(21,32,13) \oplus \underline{\bar{3}}_2(12,23,31),\\
\underline{3}_2 \otimes \underline{3}_2 &= \underline{\bar{3}}_2(32,13,21) \oplus \underline{\bar{3}}_2(23,31,12) \oplus \underline{\bar{3}}_1(22,33,11),\\
\underline{\bar{3}}_2 \otimes \underline{\bar{3}}_2 &= \underline{3}_2(32,13,21) \oplus \underline{3}_2(23,31,12) \oplus \underline{3}_1(22,33,11),\\
\underline{\bar{3}}_2 \otimes \underline{3}_2 &= \underline{3}_2 \otimes \underline{\bar{3}}_2 = 1_0(11+22+33) \oplus 1_1(11+\omega 22+\omega^2 33)\\
&\oplus\ 1_2(11+\omega^2 22+\omega 33) \oplus \underline{3}_1(23,31,12) \oplus \underline{\bar{3}}_1(32,13,21),\\
\underline{3}_1 \otimes \underline{3}_2 &= \underline{3}_2(32,13,21) \oplus \underline{\bar{3}}_2(31,12,23) \oplus \underline{3}_1(33,11,22),\\
\underline{3}_1 \otimes \underline{\bar{3}}_2 &= \underline{\bar{3}}_1(11,22,33) \oplus \underline{\bar{3}}_2(23,31,12) \oplus \underline{3}_1(21,32,13),\\
\underline{3}_2 \otimes \underline{\bar{3}}_1 &= \underline{3}_1(11,22,33) \oplus \underline{\bar{3}}_1(12,23,31) \oplus \underline{3}_2(32,13,21),\\
\underline{\bar{3}}_1 \otimes \underline{\bar{3}}_2 &= \underline{\bar{3}}_1(33,11,22) \oplus \underline{\bar{3}}_2(32,13,21) \oplus \underline{3}_2(31,12,23),
\end{aligned} \tag{A5}$$

The tensor products between singlets are obtained as

$$\begin{aligned}
&\underline{1}_0 \otimes \underline{1}_0 = \underline{1}_1 \otimes \underline{1}_2 = \underline{1}_2 \otimes \underline{1}_1 = 1_0(11),\\
&\underline{1}_1 \otimes \underline{1}_1 = \underline{1}_2(11), \quad \underline{1}_2 \otimes \underline{1}_2 = \underline{1}_1(11).
\end{aligned} \tag{A6}$$

The tensor products between triplets and singlets are obtained as

$$\underline{1}_k \otimes \underline{3}_i = \underline{3}_i(11,12,13),\ \underline{1}_k \otimes \underline{\bar{3}}_i = \underline{\bar{3}}_i(11,12,13),\ (k=0,1,2; i=1,2). \tag{A7}$$

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